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American options: the EPV pricing model★

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Summary. We explicitly solve the pricing problem for perpetual American puts and calls, and provide an efficient semi-explicit pricing procedure for options with finite time horizon. Contrary to the standard approach, which uses the price process as a primitive, we model the price process as the expected present value of a stream, which is a monotone function of a Lévy process. Certain processes exhibiting mean-reverting, stochastic volatility and/or switching features can be modeled this way. This specification allows us to consider assets that pay no dividends at all when the level of the underlying stochastic factor is too low, assets that pay dividends at a fixed rate when the underlying stochastic process remains in some range, or capped dividends.

Keywords and Phrases: Levy processes, Option pricing, Dividend paying assets.

JEL Classification Numbers: C61, D81, G12.

1 Introduction

The objective of this paper is to provide explicit pricing formulas for perpetual American options and Carr's randomization approach to American options with finite time horizon (and related real options) on assets with the dynamics more general than in the standard models. Pricing of American options is more difficult than the pricing of European ones because one needs to find not only the price but the early exercise boundary as well, that is, to solve an optimal stopping problem. Explicit analytical results for optimal stopping problems in models with the standard exponential or linear dependence on the underlying stochastic factor X_t have been known both in discrete and continuous time models with infinite time horizon, if X_t is a process with stationary independent incre-

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ments (random walk and the Brownian motion or, more generally, a Lévy process). Note that each optimal stopping problem has two inputs: a payoff, and a process which models the underlying uncertainty. From the point of view of practical applications, it is important to have tractable results for wide classes of payoffs, and the generality of the driving Lévy process is not that important. However, the bulk of the optimal stopping literature concentrated on model situations such as the pricing of call and put options in the geometric Lévy model or Lévy model, the main attention having being paid to the generality of the Lévy process involved.

Our aim is to derive analytically tractable optimal stopping rules for wide classes of payoffs and processes. As opposed to the classical option pricing theory that uses the price of an asset as the primitive, we model the payoff process as the expected present value (EPV) of a stochastic stream of payoffs, which is a monotone function of a Lévy process, under an equivalent martingale measure chosen by the market. This specification is very flexible, and includes the standard one as a special case, if the stock pays dividends (the case of non-dividend paying stock can be obtained in the limit, as the dividend rate vanishes). Our method makes it possible to price options on assets that pay no dividends at all when the value of the firm falls below a certain level, assets that pay capped dividends, and assets with several levels of the dividend rate, depending on the value of the firm or operating profits. From the point of view of economics, the definition of the value of an asset as the EPV of a stream (of revenues/dividends) is a natural one, and it turns out, that this *economically meaningful condition* allows us to derive the *optimal stopping rule in a general form*, and give a (relatively) simple and short *rigorous mathematical proof.*

Apart from generality and natural economic meaning, our approach has the following advantages over the standard approach which uses the price of an asset as the primitive. Given a candidate for the optimal exercise threshold, we calculate the option value in terms of the *expected present value operators* (EPV-operators, which calculate the EPV of a stream $g(X_t)$ under the initial process, and the EPV under its supremum and infimum processes, $\bar{X}_t = \sup_{0\le s\le t} X_s$ and $\underline{X}_t = \inf_{0\le s\le t} X_s$), and the form of the solution suggests the following description of the optimal exercise strategy. If the stream of payoffs is a decreasing function of the underlying stochastic factor (put option case), then it is optimal to exercise the option the first time the EPV of the stream of payoffs calculated for the supremum process instead of the original stochastic process becomes non-negative. Similarly, if the stream of payoffs is an increasing function of the underlying stochastic factor, then it is optimal to exercise a call-like option the first time the EPV of the stream of payoffs calculated for the infimum process instead of the original stochastic process becomes non-negative. This allows us to formulate a general optimal exercise rule: it is optimal to exercise the right for the stream of stochastic payoffs, g_t when the EPV of the stream $\underline{g}_t = \inf_{0\le s\le t} g_t$, starting at the spot level g_0 of the underlying process g_t, becomes non-negative. The interpretation of this exercise rule, in the case of the call, is as follows. Exercise is optimal when the present value of the dividends becomes sufficiently high relative to exercise price. Our rule prescribes to disregard all temporary upward movements in the dividends. If the option is exercised after a favorable shock to dividends, but later the dividends fall below the level where they used to be, it was better not to exercise the option at all, because the exercise decision is irreversible. Thus the replacement of the original dividend process with the infimum process prevents the suboptimal premature exercise of the option.

We call the above statement a *record setting bad news principle*. This principle naturally generalizes and extends Bernanke's (1983) *bad news principle* and record setting news principles spelled out in Boyarchenko (2004). In the latter paper, the principles were stated and proved for the standard calls and puts. Here the result is proved for any monotone function, $g_t = g(X_t)$ of a Lévy process. The method of the paper works for some non-monotone payoff streams as well, and the proof itself becomes fairly simple due to the systematic use of the EPV–operators.

Short review of the literature

Mc Kean (1965) calculated the price and early exercise boundary for perpetual call option in the continuous time Gaussian model, Darling et al. (1972) solved the corresponding problem in the discrete time model, for arbitrary random walk, and Merton (1973) solved the problem for the put in the continuous time Gaussian model. Starting from Gerber and Shiu (1996), a series of results for Lévy processes of varying degree of generality were obtained by various authors, using different methods [(see the bibliography in Boyarchenko and Levendorskiǐ (2000, 2002a,b), Mordecki (2002a), Alili and Kyprianou (2004)]. In Boyarchenko and Levendorskiǐ (2000, 2002a,b), the early exercise boundaries for perpetual American put and call options were calculated for wide classes of Lévy processes used in empirical studies of financial markets, including diffusions with exponentially distributed jumps (used by Duffie et al., 2000, in affine term structure models), Hyperbolic processes, Normal Inverse Gaussian processes (see Eberlein et al., 1998; Barndorff-Nielsen, 1998, respectively), and the Koponen (1995) model and its extension constructed in Boyarchenko and Levendorskiǐ (2000). Later, the extended Koponen model was used under the name KoBoL family in Boyarchenko and Levendorskiǐ (2002a,b,c), and under the name CGMY model in Carr et al. (2002). The form of the result in Boyarchenko and Levendorskiǐ (2000, 2002a,b) makes sense for any Lévy processes, and later Mordecki (2002a) obtained the result for any Lévy process. However, the proof of one of the key points was missing; for the missing part, see Mordecki (2004). An independent proof for general Lévy processes was obtained by Alili and Kyprianou (2004).

The standard approach uses the direct link to the first passage problem for a Lévy process or random walk, and it can be directly applied for standard payoffs of the put and call options in the geometric Lévy model or Lévy model (see e.g., Mordecki (2002a,b)); each type of payoffs should be treated individually, and it is difficult to adjust the standard approach to payoffs of a general form. Further, the geometric Lévy model does not fit empirical data well, and to improve the fit, a process should exhibit mean-reversion, stochastic volatility, and/or switching features – see Bates (2003), Chernov et al. (2003) and the bibliography therein.

Unfortunately, standard mean-reverting, stochastic volatility and switching models lead to analytically efficient pricing formulas for contingent claims of the European type only (see, e.g., Duffie et al., 2000), and to the lesser extent, for options with a fixed barrier. For American options under mean reverting and stochastic volatility processes, explicit formulas are not known, and standard switching models lead to systems of equations with several unknown functions – values of an option in each state. These models are analytically untractable for general Lévy processes, and even in the Brownian motion case lead to highly non-linear systems. The results of this paper partially fill in the gap between the Lévy model and more general models by constructing analytically tractable one-factor models, which can exhibit any of these features.

In Boyarchenko and Levendorskiǐ (2000, 2002a,b), the results for standard payoffs were obtained as a byproduct of a general result for more general payoffs. We used the reduction to the free boundary problem and standard analytical argument, which required a certain regularity of a Lévy process. In the present paper, we simplify our method, and use a hybrid of probabilistic and analytical argument to consider more general Lévy processes, and much more general dependence of the payoff on the stochastic factor X_t. For a discrete time analog, see Boyarchenko and Levendorskiǐ (2004).

In addition, here we develop an efficient pricing scheme for American options with finite time horizon. We use the variant of analytic method of lines (Carr and Faguet, 1994) or equivalently, Carr's randomization procedure (Carr, 1998), which is, essentially, a sequence of embedded perpetual options. Even in this situation, with arbitrarily many embedded options, the method remains analytically tractable. In Boyarchenko and Levendorskiǐ (2002b, Ch. 6) and Boyarchenko and Levendorskiǐ (2004), less general class of Lévy processes and the standard American put were considered. In the case of the Brownian motion with embedded exponentially distributed jumps, and the stream of payoffs modeled as piece-wise exponential polynomials of the underlying process, the solution of the optimal stopping problems in the sequence reduces to a (longer) sequence of integrals of a simple structure. Using the method of indeterminate coefficients, we reduce the calculations to systems of linear (algebraic) equations. In addition, on each time step, it is necessary to find a unique zero of a monotone continuous function. This gives a very fast and accurate numerical procedure. A simpler version of the same procedure can be applied to price options with finite time horizon and a fixed barrier (barrier options).

The rest of the paper is organized as follows. In Section 2, we give examples of dividend streams, which can be treated in the framework of our approach, and formulate the optimal stopping problem. In Section 3, we explain how our method works in the Gaussian model, for call-like options, using an elementary technique. In Section 4, we introduce the EPV operators, and formulate the main results for perpetual calls and puts. The proofs are given in Section 6 and Section 7, with most technical parts delegated to Section 9. In Section 5, we recall the basic definitions of the theory of Lévy processes, calculate action of the EPV operators for the case of diffusions with exponentially distributed jumps, and give explicit realizations of the pricing rules formulated in Section 4. Carr's randomization is considered in Section 8. Section 10 concludes.

2 Perpetual American options: general set-up

Payoff specification

Assume that the underlying stochastic factor $\{X_t\}$ is a Lévy process under a risk-neutral measure chosen by the market, and denote the corresponding expectation operator by E. The EPV of a stream $g(X_t)$, at the spot level $X_0 = x$, is given by

$$(U_X^q g)(x) := E^x \left[\int_0^\infty e^{-qt} g(X_t) dt \right], \tag{2.1}$$

where $q > 0$ is the riskless rate. For the perpetual American call on an asset, which pays the revenue/dividend stream δ_t, the payoff function is

$$G(x) = (U_X^q \delta)(x) - K = U_X^q(\delta(\cdot) - qK)(x),$$

where K is the strike price, and for the perpetual American put, the payoff function is

$$G(x) = K - (U_X^q \delta)(x) = U_X^q(qK - \delta(\cdot))(x).$$

Notice that we use $G(X_t)$ rather than $\max\{G(X_t), 0\}$, which is admissible because it is not optimal to exercise the option unless $G(X_t)$ is positive (the equivalence of these two specifications was used in Darling et al. (1972)). The standard specification of the payoff for the call is $G(X_t) = e^{X_t} - K$, where X_t is the log-price of the stock, and it can be represented as the EPV of a certain stream as follows. Set $\Psi(z) = \ln E\left[e^{zX_1}\right]$ (this is the Lévy exponent of a Lévy process X_t). If $\Psi(1) < q$, which implies that we deal with a dividend-paying asset, then the standard payoff can be represented as the EPV of the stream $g(X_t) = (q - \Psi(1))e^{X_t} - qK$. Similar representation obtains for the put option on a dividend-paying stock.

Examples of admissible streams of payoffs and dividend streams

1. The asset pays no dividends when the underlying stochastic factor (say, the value of the firm) is too low: $\delta(x) = 0,\ x \le x_0$, and dividends increase exponentially after the factor crosses a certain level:
$$\delta(x) = (e^x - e^{x_0})^+ := \max\{0, e^x - e^{x_0}\}. \tag{2.2}$$
2. The asset pays no dividends when the underlying stochastic factor is too low: $\delta(x) = 0,\ x \le x_0$, and the dividends increase not so fast after the factor crosses a certain level:
$$\delta(x) = (x - x_0)^+. \tag{2.3}$$
3. The asset pays no dividends when the underlying stochastic factor is too low: $\delta(x) = 0,\ x \le x_0$. When the critical level is crossed, the dividends increase but eventually the growth slows and essentially stops (capped dividends):
$$\delta(x) = (e^{-x_0} - e^{-x})^+. \tag{2.4}$$

4. The asset pays dividends at a fixed rate when the underlying process is within a certain range; when the process arrives in the next range, the dividend rate changes by a jump:
$$\delta(x) = \sum_j \delta_j \mathbf{1}_{[d_j,+\infty)}(x), \tag{2.5}$$
where $\mathbf{1}_{[a,+\infty)}$ is the indicator function of $[a,+\infty)$. The sum can be finite (in this case, the dividends are capped, as in Example 3) or infinite, which allows for unbounded growth of dividends. If the intervals on which δ is constant are large, this model can be viewed as a switching model: when X_t remains inside one of these intervals, the fluctuations of $G(X_t) = U^q_X g(X_t)$ are relatively small, but the differences among levels of $G(X_t)$ for different intervals can be large.
5. If the payoff stream $g(X_t)$ is a convex function of X_t for $X_t < x_0$, and concave for $X_t > x_0$, the process X_t may exhibit a mean-reverting feature. A simple example: $g(X_t) = -1 + e^{X_t}$ for $X_t \le 0$, and $g(X_t) = 1 - e^{-X_t}$ for $X_t \ge 0$.
6. The dividends are paid in a constant proportion to the firm's value: $\delta(X_t) = \lambda X_t$, but the value itself is an increasing function of a Lévy process, Y_t: $X_t = f(Y_t)$.
7. We can easily generalize examples 1–5 by using $\delta(f(Y_t))$, where f is an increasing function, and Y_t is a Lévy process.

Optimal stopping problem

The rational price of the option with the payoff $G(X_t)$ is given by
$$V(x) = \sup E^x[e^{-q\tau} G(X_\tau)], \tag{2.6}$$
where the supremum is taken over a set $\mathcal{M}$ of all stopping times $\tau = \tau(\omega)$ satisfying $0 \le \tau(\omega) \le +\infty$, $\omega \in \Omega$; if $\tau(\omega) = +\infty$, then $G(\tau(\omega)) = 0$ by definition (see, e.g., Shiryaev (1999), XVIII, 2). The payoff is modeled as $G(X_t) = U^q_X g(X_t)$, where g is an arbitrary monotone function; certain non-monotone g are admissible as well. If g is not continuous, we impose an additional weak condition on the process X_t. The optimal stopping time, τ, turns out to be the hitting time of a semi-infinite interval of the form $(-\infty, h]$ (put options) or $[h, +\infty)$ (call options). We denote these hitting times by τ^-_h and τ^+_h, respectively.

3 Perpetual American options: solution in the Gaussian case

In this section, we explain our method in the case of Gaussian process, assuming that in the case of a call-like option, the optimal stopping rule is of the form: exercise the option when the price reaches a certain threshold (early exercise boundary). We consider the perpetual call option on the stock with the dividend stream $\delta(X_t)$ and strike K, under the following conditions:

(i) $\delta(X_t)$ is a non-decreasing function of an underlying stochastic factor X_t;
(ii) X_t is a Gaussian process with the dynamics

$$dX_t = \mu X_t + \sigma dW_t, \tag{3.7}$$

where $\mu \in \mathbf{R}$, $\sigma > 0$, and dW_t is the increment of the standard Brownian motion with 0 drift and unit variance;
(iii) function $g(X_t) := \delta(X_t) - qK$ is negative for sufficiently low levels of X_t, and positive for sufficiently large levels of X_t.

Let h be a candidate for the optimal exercise boundary. Then at the current level $x = X_t$ of the underlying stochastic factor, the option price is

$$\begin{aligned} V(x;h) &= E^x\left[e^{-q\tau_h^+}(P(X_{\tau_h^+}) - K)\right] & (3.8)\\ &= E^x\left[e^{-q\tau_h^+} U_X^q g(X_{\tau_h^+})\right] \\ &= E^x\left[e^{-q\tau_h^+} E^{X_{\tau_h^+}}\left[\int_0^{+\infty} e^{-qt} g(X_t)dt\right]\right] \\ &= E^x\left[\int_{\tau_h^+}^{+\infty} e^{-qt} g(X_t)dt\right] \\ &= v_{+\infty}(x) - v_h(x), & (3.9) \end{aligned}$$

where

$$v_{+\infty}(x) = E^x\left[\int_0^{+\infty} e^{-qt} g(X_t)dt\right] = U_X^q g(x), \tag{3.10}$$

$$v_h(x) = E^x\left[\int_0^{\tau_h^+} e^{-qt} g(X_t)dt\right]. \tag{3.11}$$

We calculate $v_{+\infty}(x) = U_X^q g(x)$, next, $v_h(x)$, and then find a convenient form for $V(x;h)$, which will allow us to guess the optimal h quite easily.
Calculation of $U_X^q g(x)$. As it is well-known, the EPV of the perpetual stream g is the solution of the Black-Scholes equation for the Brownian motion model:

$$\frac{\sigma^2}{2}V''(x) + \mu V'(x) - qV(x) + g(x) = 0, \tag{3.12}$$

which grows at infinity not faster than the stream $g(x)$. To find the solution of (3.12) in the form, which is convenient for our approach, we write down (3.12) in the form

$$\left(-\frac{\sigma^2}{2}\partial^2 - \mu\partial + q\right)V(x) = g(x), \tag{3.13}$$

where ∂ is the differentiation operator which maps a function V into $\partial V = V'$ (hence, ∂^2 maps V into V''). Next, we factorize the operator

$$A(\partial) := q - \mu\partial - \frac{\sigma^2}{2}\partial^2$$

in the LHS of (3.13):

$$A(\partial) = qA^-A^+ := q \cdot \frac{\beta^- - \partial}{\beta^-} \cdot \frac{\beta^+ - \partial}{\beta^+},$$

where $\beta^- < 0 < \beta^+$ are the roots of the "fundamental quadratic" equation in the terminology of Dixit and Pindyck (1996):

$$q - \mu\beta - \frac{\sigma^2}{2}\beta^2 = 0,$$

and rewrite equation (3.13) in the form

$$A^-A^+qV = g. \tag{3.14}$$

We can find the normalized solution qV in two steps. First, we find g_2, the solution of the equation

$$A^-g_2 = g. \tag{3.15}$$

When this solution, denote it $\mathcal{E}^-g$, is found, we solve the equation

$$A^+qV = g_2. \tag{3.16}$$

Denote the solution of the equation (3.16) by $\mathcal{E}^+g_2$. Then we can write

$$qU_X^q g(x) = qV(x) = (\mathcal{E}^+\mathcal{E}^-g)(x). \tag{3.17}$$

Equation (3.15) is equivalent to

$$\beta^-g_2(x) - g_2'(x) = \beta^-g(x). \tag{3.18}$$

This is an ODE of the first order, with constant coefficients. Assume that g is bounded as $x \to -\infty$. Then (3.18) has a unique solution which is bounded as $x \to -\infty$:

$$g_2(x) = (-\beta^-)\int_{-\infty}^0 e^{-\beta^-y}g(x+y)dy. \tag{3.19}$$

(This is a standard fact from the theory of ODE; the reader can easily verify (3.19) by substituting (3.19) into (3.18) and integrating by parts.) Thus,

$$g_2(x) = \mathcal{E}^-g(x) = E[x + Y^-], \tag{3.20}$$

where Y^- is the exponential random variable on $\mathbf{R}_- := (-\infty, 0]$ with the mean $1/\beta^-$ (and PDF $p^-(y) = -\beta^-e^{-\beta^-y}$). Similarly, equation (3.16) for $u := qV$ is equivalent to

$$\beta^+u(x) - u'(x) = \beta^+g_2(x). \tag{3.21}$$

This is an ODE of the first order, with constant coefficients. Assume that g increases slower than $e^{\beta^+ x}$ as $x \to +\infty$; then (3.21) has a unique solution which increases slower than $e^{\beta^+ x}$, as $x \to +\infty$:

$$u(x) = \beta^+ \int_0^{+\infty} e^{-\beta^+ y} g_2(x+y)dy. \tag{3.22}$$

Thus,

$$qV(x) = \mathcal{E}^+ g_2(x) = E[x + Y^+], \tag{3.23}$$

where Y^+ is the exponential random variable on $\mathbf{R}_+ := [0, +\infty)$ with the mean $1/\beta^+$ (and PDF $p^+(y) = \beta^+ e^{-\beta^+ y}$).

Calculation of $v_h(x)$ requires more work. First, v_h is the bounded solution of the equation (3.12) but on the interval $(-\infty, h)$ only. On $[h, +\infty)$, $v_h(x) = 0$. We can write (3.12) as an equation on the whole real line after we introduce an auxiliary function f that vanishes on $(-\infty, h]$:

$$A(\partial)v_h(x) = g(x) + f(x), \quad \forall\, x, \tag{3.24}$$

or equivalently,

$$A^- A^+ q v_h(x) = g(x) + f(x), \quad \forall\, x. \tag{3.25}$$

Similarly to (3.19)–(3.20), we find

$$\begin{aligned} A^+ q v_h(x) &= \mathcal{E}^- g(x) + \mathcal{E}^- f(x) && (3.26)\\ &= E[g(x+Y^-)] + E[f(x+Y^-)]. && (3.27) \end{aligned}$$

Since $f(x) = 0$ for all $x \le h$, and Y^- is a distribution on $\mathbf{R}_-$, the rightmost term in the RHS of (3.27) vanishes on $(-\infty, h]$. Hence, $u_h(x) = q v_h(x)$ is the solution to the equation

$$\beta^+ u_h(x) - u_h'(x) = \beta^+ \mathcal{E}^- g(x) \tag{3.28}$$

on $(-\infty, h)$, that vanishes on $[h, +\infty)$. This solution is

$$\begin{aligned} u_h(x) &= \beta^+ \int_0^{h-x} e^{-\beta^+ y} (\mathcal{E}^- g)(x+y)dy \\ &= \beta^+ \int_0^{+\infty} e^{-\beta^+ y} \mathbf{1}_{(-\infty,h]}(x+y)(\mathcal{E}^- g)(x+y)dy. && (3.29) \end{aligned}$$

Thus,

$$q v_h(x) = (\mathcal{E}^+ \mathbf{1}_{(-\infty,h]} \mathcal{E}^- g)(x). \tag{3.30}$$

Calculation of $V(x;h)$*, and the optimal choice of* h. Now we can find

$$\begin{aligned} qV(x;h) &= (\mathcal{E}^+ \mathcal{E}^- g)(x) - (\mathcal{E}^+ \mathbf{1}_{(-\infty,h]} \mathcal{E}^- g)(x) \\ &= (\mathcal{E}^+ \mathbf{1}_{(h,+\infty)} \mathcal{E}^- g)(x) && (3.31)\\ &= E[\mathbf{1}_{(h,+\infty)}(x+Y^+) w(x+Y^+)], && (3.32) \end{aligned}$$

where $w(x) = \mathcal{E}^- g(x) = E[g(x+Y^-)]$. Under the standing assumptions on g, w is a continuous increasing function, which changes sign. Hence, the expectation in (3.32) is maximal if and only if the multiplication by the indicator function $\mathbf{1}_{(h,+\infty)}$ cuts off all the negative values of w, and leaves positive ones intact. It follows that the optimal choice of h is the solution to the equation

$$w(x) := \mathcal{E}^- g(x)(= E[g(x+Y^-)]) = 0. \tag{3.33}$$

When h is found, the rational option price is calculated from (3.32):

$$V(x;h) = q^{-1}(\mathcal{E}^+ \mathbf{1}_{(h,+\infty)} \mathcal{E}^- g)(x). \tag{3.34}$$

Example. Let the dividend stream be given by $\delta(x) = (e^x - e^{x_0})_+$, and let K be the strike price. Assume that $\beta^+ > 1$. Then the payoff $G(x) = U_X^q g(x)$, where $g(x) = \delta(x) - qK$, is finite. Since $g(x) < 0$ for $x \le x_0$, we have $w(x) = E[g(x+Y^-)] < 0$ for these x. Therefore, the optimal h is on the interval $x > x_0$. For $x > x_0$,

$$\begin{aligned}
w(x) &= \mathcal{E}^- g(x) \\
&= (-\beta^-) \int_{-\infty}^0 e^{-\beta^- y}(\delta(x+y) - qK)dy \\
&= (-\beta^-) \int_{x_0 - x}^0 e^{-\beta^- y}(e^{x+y} - e^{x_0})dy - qK \\
&= Ae^x + Be^{\beta^- x} - C,
\end{aligned}$$

where $A = -\beta^-/(1-\beta^-), B = e^{(1-\beta^-)x_0}/(1-\beta^-), C = e^{x_0} + qK$, therefore h is the root of the equation

$$Ae^x + Be^{\beta^- x} - C = 0$$

on the interval $(x_0, +\infty)$. Since A and B are positive, function $\Phi(x) = Ae^x + Be^{\beta^- x} - C$ is concave, and the root can easily be found by any numerical method, with any accuracy. After h is found, we calculate the option value using (3.34): for $x < h$,

$$\begin{aligned}
V(x;h) &= q^{-1}\beta^+ \int_0^{+\infty} e^{-\beta^+ y} \mathbf{1}_{(h,+\infty)}(x+y)(Ae^{x+y} + Be^{\beta^-(x+y)} - C)dy \\
&= q^{-1}\beta^+ \int_{h-x}^{+\infty} e^{-\beta^+ y}(Ae^{x+y} + Be^{\beta^-(x+y)} - C)dy \\
&= e^{\beta^+(x-h)} q^{-1}\beta^+ \int_0^{+\infty} e^{-\beta^+ y}(Ae^{h+y} + Be^{\beta^-(h+y)} - C)dy \\
&= D(h)e^{\beta^+(x-h)},
\end{aligned}$$

where

$$D(h) = q^{-1}\beta^+ \int_0^{+\infty} e^{-\beta^+ y}(Ae^{h+y} + Be^{\beta^-(h+y)} - C)dy$$
$$= \frac{Ae^h\beta^+}{q(\beta^+ - 1)} + \frac{Be^{\beta^- h}\beta^+}{q(\beta^+ - \beta^-)} - \frac{C}{q}.$$

4 Perpetual American options: solution in the Lévy case

In this section, we present the solution to the optimal stopping problem for a wide class of Lévy processes satisfying the (ACP)–property (absolute continuity of potential kernels: see, e.g., Bertoin, 1996; or Sato, 1999). The (ACP) property is fairly weak; in particular, if the transition kernel has a density, then this property holds. The results are formulated in terms of the EPV operators $U^q_{\bar{X}}$ and $U^q_{\underline{X}}$, of the supremum and infimum processes $\bar{X}_t = \sup_{0\le s\le t} X_s$ and $\underline{X}_t = \inf_{0\le s\le t} X_s$ of X_t. The EPV-operators $U^q_{\bar{X}}$ and $U^q_{\underline{X}}$ are defined by

$$U^q_{\bar{X}} g(x) := E^x\left[\int_0^\infty e^{-qt} g(\bar{X}_t)dt\right] := E\left[\int_0^\infty e^{-qt} g(\bar{X}_t)dt \mid X_0 = x\right]$$

and

$$U^q_{\underline{X}} g(x) := E^x\left[\int_0^\infty e^{-qt} g(\underline{X}_t)dt\right] := E\left[\int_0^\infty e^{-qt} g(\underline{X}_t)dt \mid X_0 = x\right],$$

respectively. Explicit analytic formulas for diffusions with exponentially distributed jumps will be given in Section 5 (for the general case, see Boyarchenko and Levendorskiǐ, 2002a,b). In the limit, when the jump component vanishes, the formulas for $U^q_{\bar{X}}$ and $U^q_{\underline{X}}$ give $qU^q_{\bar{X}} = \mathcal{E}^+$, $qU^q_{\underline{X}} = \mathcal{E}^-$, where $\mathcal{E}^\pm$ are operators introduced in Section 3.

Perpetual call options. This is the case of a non-decreasing $G = U^q_{\bar{X}} g$. To ensure that G were finite, we impose the condition

$$|g(x)| \le C(e^{\sigma_+ x} + e^{\sigma_- x}), \quad \forall\, x, \tag{4.35}$$

where $\sigma_- \le 0 < \sigma_+$ satisfy the condition

$$E\left[\int_0^{+\infty} e^{-qt+\sigma X_t}dt\right] < +\infty, \quad \sigma = \sigma_\pm. \tag{4.36}$$

In terms of the Lévy exponent of a Lévy process, an equivalent condition is

$$q - \Psi(\sigma_\pm) > 0. \tag{4.37}$$

The optimal stopping problem for the call option is trivial if the supremum process is trivial. Hence, we presume that the supremum process is non-trivial. In Section 7, we prove the optimality in the class $\mathcal{M}$, under a weak condition that the stream g is a non-decreasing function. (This condition is not necessary; in fact, the proof in the paper works in some situations when the stream is non-monotone.) Since a non-decreasing g is measurable, and X satisfies the (ACP)-property, the payoff $G(X_t)$ is a continuous function of the factor X_t. Let $w := U^q_{\underline{X}} g$ be the EPV of the stream g under the infimum process. Then w is non-decreasing; if g is increasing, then w is increasing as well. The crucial condition for our method is that w must have a zero, and then this zero is the early exercise boundary. Assume that w changes sign; sufficient conditions are

$$g(-\infty) < 0, \quad g(+\infty) > 0 \tag{4.38}$$

(one or both limits may be infinite). If w is continuous (a sufficient condition is that g is continuous), then the equation

$$w(x) := U^q_{\underline{X}} g(x) = 0 \tag{4.39}$$

has a root, call it h^*. Since g is monotone and satisfies (4.38), w may be locally constant only in a neighborhood of $-\infty$, where it is negative; outside this neighborhood, w is increasing, and a zero of (4.39) is unique. In Section 7, we will show that the optimal stopping time is $\tau^+_{h^*}$, and the rational call price is

$$V^*(x) = qU^q_{\bar{X}} \mathbf{1}_{[h^*,+\infty)} U^q_{\underline{X}} g(x). \tag{4.40}$$

In the Gaussian case, $qU^q_{\bar{X}} = \mathcal{E}^+$, $qU^q_{\underline{X}} = \mathcal{E}^-$, and the RHS in (4.40) equals the RHS in (3.34). The proof for more general Lévy processes is more involved. For explicit realizations of formulas (4.39) and (4.40), see Section 5.

For a general monotone g and process X_t satisfying the (ACP)-property, it is possible that w is discontinuous, and a zero does not exist; generically (w.r.t. to the strike price, say) it does exist. So, if g is discontinuous, we assume that X_t enjoys the *call-regularity* property

(CR) *$U^q_{\underline{X}} g$ is continuous for any non-decreasing g satisfying (4.35), and $q > 0$.*

The same arguments as in Boyarchenko and Levendorskiĭ (2002a,b) show that if the smooth pasting condition for the put option holds, then the (CR) condition holds as well. In particular, it holds if the gaussian component of the process X_t is non-trivial. For further analysis of the smooth pasting condition, see Alili and Kyprianou (2004).

Perpetual put options. This is the case of a non-increasing $G = U^q_{\bar{X}} g$. To ensure that G were finite, we impose the condition (4.35). This condition implies that the stock pays dividends. The result for the put on a stock which pays no dividends can be deduced as the limit of the result for a dividend-paying option, when the dividend flow vanishes. We also require that g is a non-increasing function satisfying

$$g(-\infty) > 0, \quad g(+\infty) < 0 \tag{4.41}$$

(one or both limits may be infinite). The optimal stopping problem for a put-like option is trivial if the infimum process is trivial. Hence, we presume that the infimum process is non-trivial. If $w = U^q_{\bar{X}} g$ is continuous (a sufficient condition is that g is continuous), then the equation

$$w(x) = 0, \tag{4.42}$$

has a unique root, call it h_*, and the optimal stopping time is $\tau^-_{h_*}$. In Section 6, we will show that the rational put price is

$$V_*(x) = qU^q_{\underline{X}}\mathbf{1}_{(-\infty,h_*]}U^q_{\bar{X}}g(x). \tag{4.43}$$

If g is discontinuous, we assume that X_t enjoys the *put regularity* property

(PR) $U^q_{\bar{X}}g$ *is continuous for any non-increasing* g *satisfying (4.35), and* $q > 0$.

A sufficient condition is: the smooth pasting condition for the call option is valid. In particular, (PR) holds if the gaussian component of the process X_t is non-trivial.

5 Lévy processes and explicit pricing formulas

General definitions. Recall that a Lévy process is a process with stationary independent increments (for general definitions of the theory of Lévy processes, see, e.g., Bertoin, 1996; Sato, 1999). A Lévy process may have a Gaussian component and/or pure jump component. The latter is characterized by the density of jumps, which is called the Lévy density. We denote it by $F(dx)$. If X_t is a Lévy process with finite variation jump component, then the Lévy exponent is given by

$$\Psi(z) = bz + \frac{\sigma^2}{2}z^2 + \int_{-\infty}^{+\infty}(e^{zy}-1)F(dy), \tag{5.44}$$

where σ^2 and b are the variance and drift of the Gaussian component, and $F(dy)$ satisfies $\int_{\mathbf{R}\setminus\{0\}}\min\{1,|y|\}F(dy) < +\infty$. Equation (5.44) is a special case of the Lévy-Khintchine formula; for the general case, see, e.g., Sato (1999). In this paper, we will illustrate our general results for the case of the Brownian motion with embedded jumps used in Duffie et al. (2000). The Lévy density is

$$F(dx) = c^+\lambda^+e^{-\lambda^+x}\mathbf{1}_{(0,+\infty)}(x)dx + c^-(-\lambda^-)e^{-\lambda^-x}\mathbf{1}_{(-\infty,0)}(x)dx, \tag{5.45}$$

where $\lambda^+ > 0 > \lambda^-$, and $c_\pm > 0$, and the Lévy exponent is

$$\Psi(z) = \frac{\sigma^2}{2}z^2 + bz + \frac{c^+z}{\lambda^+-z} + \frac{c^-z}{\lambda^--z}. \tag{5.46}$$

Equation (5.45) means that during an infinitesimal time interval dt, the probability of upward jumps is c^+dt. If the upward jump has happened, the probability of a jump from the current level x into an interval $[x+y, x+y+dy]$ is $\lambda_+e^{-\lambda_+y}dy$. The negative part of the Lévy density admits a similar interpretation.

EPV-operators. The Lévy exponent appears when we calculate the action of the infinitesimal generator of X_t, denoted L, on exponential functions: $Le^{zx} = \Psi(z)e^{zx}$. The operator (2.1) calculates the expected present value of a stream. From the fundamental relation between the infinitesimal generator and the resolvent,

$$(q-L)U^q_X = U^q_X(q-L) = I, \tag{5.47}$$

one concludes that U^q_X acts on exponential functions as the multiplication operator by the number $(q-\Psi(z))^{-1}$:

$$U^q_X e^{zx} = (q-\Psi(z))^{-1} e^{xz}. \tag{5.48}$$

One can also derive (5.48) from (2.1) with $g(x)=e^{zx}$, using the definition $E[e^{zX_t}]=e^{t\Psi(z)}$ under the integral sign and evaluating the integral. Similarly, it is straightforward to check that the EPV operators $qU^q_{\bar{X}}$ and $qU^q_{\underline{X}}$ also act on an exponential function e^{zx} as multiplication operators by certain numbers, which we denote $\kappa^+_q(z)$ and $\kappa^-_q(z)$, respectively:

$$qU^q_{\bar{X}} e^{zx} = \kappa^+_q(z)e^{zx}, \quad qU^q_{\underline{X}} e^{zx} = \kappa^-_q(z)e^{zx}. \tag{5.49}$$

These numbers are

$$\kappa^+_q(z) = qE\left[\int_0^\infty e^{-qt}e^{z\bar{X}_t}dt \mid X_0=0\right], \tag{5.50}$$

$$\kappa^-_q(z) = qE\left[\int_0^\infty e^{-qt}e^{z\underline{X}_t}dt \mid X_0=0\right]. \tag{5.51}$$

Notice that $\kappa^+_q(z)$ (resp., $\kappa^-_q(z)$) is analytic on the half-plane $\Re z<0$ (resp., $\Re z>0$), and continuous up to the boundary. The Wiener-Hopf factorization formula reads (see, e.g., Sato (1999), Sect. 45)

$$\frac{q}{q-\Psi(z)} = \kappa^+_q(z)\kappa^-_q(z). \tag{5.52}$$

Applying U^q_X, $U^q_{\bar{X}}$ and $U^q_{\underline{X}}$ to an exponential function $g(x)=e^{zx}$ and using (5.48) and (5.49)–(5.52), we obtain

$$U^q_X g(x) = qU^q_{\bar{X}}U^q_{\underline{X}}g(x) = qU^q_{\underline{X}}U^q_{\bar{X}}g(x). \tag{5.53}$$

By linearity, (5.53) holds for linear combinations of exponents, and integrals of exponents, hence, for wide classes of functions. Equation (5.53) means that the EPV-operator of a Lévy process admits a factorization into a product of the EPV-operators of the supremum and infimum processes.

Diffusions with exponentially distributed jumps. Let X_t be a process with the characteristic exponent (5.46). Then $q-\Psi(z)$ is a rational function, which has 4 real zeroes; two of them are positive, and two negative. We will call them β^-_j and β^+_j, $j=1,2$, respectively. It is easy to show that λ^- separates the negative roots, and λ^+ – the positive ones: $\beta^-_2<\lambda^-<\beta^-_1<0<\beta^+_1<\lambda^+<\beta^+_2$. Since $q-\Psi(z)$ is rational, the factors $\kappa^\pm(z)$ can easily be obtained by representing the LHS in (5.52) as the fraction of two polynomials, factorizing these polynomials out, and collecting the factors with positive (respectively, negative) zeroes. For details of these calculations and calculations below, see, e.g., Levendorskiĭ (2004). We obtain

$$\kappa^+_q(z) = \frac{\beta^+_1\beta^+_2(\lambda^+-z)}{(\beta^+_1-z)(\beta^+_2-z)\lambda^+} = \sum_{j=1,2}\frac{a^+_j}{\beta^+_j-z}, \tag{5.54}$$

where

$$a_1^+ = \frac{\beta_1^+\beta_2^+(\lambda^+ - \beta_1^+)}{(\beta_2^+ - \beta_1^+)\lambda^+}, \quad a_2^+ = \frac{\beta_1^+\beta_2^+(\lambda^+ - \beta_2^+)}{(\beta_1^+ - \beta_2^+)\lambda^+}, \tag{5.55}$$

are positive, and

$$\kappa_q^-(z) = \frac{\beta_1^-\beta_2^-(\lambda^- - z)}{(\beta_1^- - z)(\beta_2^- - z)\lambda^-} = \sum_{j=1,2} \frac{a_j^-}{\beta_j^- - z}, \tag{5.56}$$

where

$$a_1^- = \frac{\beta_1^-\beta_2^-(\lambda^- - \beta_1^-)}{(\beta_2^- - \beta_1^-)\lambda^-}, \quad a_2^- = \frac{\beta_1^-\beta_2^-(\lambda^- - \beta_2^-)}{(\beta_1^- - \beta_2^-)\lambda^-} \tag{5.57}$$

are negative. The operators $qU^q_{\bar{X}}$ and $qU^q_{\underline{X}}$ act as follows:

$$qU^q_{\bar{X}}u(x) = \sum_{j=1,2} a_j^+ \int_0^\infty e^{-\beta_j^+ y} u(x+y)dy, \tag{5.58}$$

$$qU^q_{\underline{X}}u(x) = \sum_{j=1,2} (-a_j^-) \int_{-\infty}^0 e^{-\beta_j^- y} u(x+y)dy. \tag{5.59}$$

To see this, it suffices to insert $u(x) = e^{zx}$, and use the definition of the numbers $\kappa_q^+(z), \kappa_q^-(z)$. If u is a piece-wise exponential polynomial, then the calculation of the integrals in (5.58)–(5.59) is straightforward. Using the method of indeterminate coefficients, these calculations can be reduced to systems of linear algebraic equations. For details, see Levendorskiǐ (2004).

Explicit formulas for the perpetual call: the case of exponential jump-diffusions. As an example, consider the call option on a stock which pays dividends at rate $\delta(X_t)$, where δ is a non-decreasing function satisfying $\delta(+\infty) > qK > \delta(-\infty)$. We use (5.59), and conclude that h^* is the solution to the equation

$$q^{-1} \sum_{j=1,2} (-a_j^-) \int_{-\infty}^0 e^{-\beta_j^- y} \delta(x+y)dy - K = 0, \tag{5.60}$$

which can easily be found by standard numerical methods. In particular, in Examples 1–5 of Section 2 (and in many others), the integral in (5.60) can be calculated explicitly, and we have to solve an algebraic equation. Consider, for instance, Example 1 with $x_0 = 0$. For $x \le 0$, the LHS in (5.60) is $-K$, therefore a unique zero is on the positive half-axis. For $x > 0$, we change the variables

$$q^{-1} \sum_{j=1,2} (-a_j^-) e^{\beta_j^- x} \int_{-\infty}^x e^{-\beta_j^- y} \delta(y)dy - K = 0, \tag{5.61}$$

and calculate

$$\int_{-\infty}^x e^{-\beta_j^- y}\delta(y)dy = \int_0^x e^{-\beta_j^- y}(e^y - 1)dy = \frac{e^{(1-\beta_j^-)x} - 1}{1 - \beta_j^-} - \frac{e^{-\beta_j^- x} - 1}{-\beta_j^-}.$$

We see that the first term on the RHS of (5.61) is

$$\sum_{j=1,2}\frac{(-a_j^-)}{q}\left\{\frac{e^x-e^{-\beta_j^- x}}{1-\beta_j^-}-\frac{1-e^{-\beta_j^- x}}{-\beta_j^-}\right\}$$
$$=q^{-1}\sum_{j=1,2}\left\{\frac{-a_j^- e^x}{1-\beta_j^-}-\frac{a_j^-}{\beta_j^-}+\frac{a_j^- e^{-\beta_j^- x}}{(1-\beta_j^-)\beta_j^-}\right\}$$
$$=q^{-1}\left\{\kappa_q^-(1)e^x-1+\sum_{j=1,2}\frac{a_j^- e^{-\beta_j^- x}}{(1-\beta_j^-)\beta_j^-}\right\}.$$

Thus, the equation for h^* is

$$\kappa_q^-(1)e^x+\sum_{j=1,2}\frac{a_j^-}{(1-\beta_j^-)\beta_j^-}e^{-\beta_j^- x}=qK+1,\quad x>0,$$

and it can be solved quite easily (numerically). The equation being solved, we represent $w=U^q_{\underline{X}}\delta-K$ in (4.39) in the form

$$w(x)=q^{-1}\sum_{j=1,2}(-a_j^-)\int_{-\infty}^0 e^{-\beta_j^- y}(\delta(x+y)-\delta(h^*+y))dy,$$

and calculate the rational call price using (5.58):

$$\begin{aligned}
V^*(x) &= \sum_{k=1,2}a_k^+\int_0^{+\infty}e^{-\beta_k^+ y}\mathbf{1}_{[h^*,+\infty)}(x+y)w(x+y)dy\\
&= \sum_{k=1,2}a_k^+\int_{h^*-x}^{+\infty}e^{-\beta_k^+ y}w(x+y)dy\\
&= \sum_{j,k=1,2}\frac{a_k^+(-a_j^-)}{q}\int_{h^*-x}^{+\infty}\int_{-\infty}^0 e^{-\beta_k^+ y-\beta_j^- z}[\delta(x+y+z)-\delta(h^*+z)]dzdy\\
&= q^{-1}\sum_{j,k=1,2}a_k^+(-a_j^-)e^{\beta_k^+(x-h^*)}\\
&\quad\cdot\int_0^{+\infty}\int_{-\infty}^0 e^{-\beta_j^- z-\beta_k^+ y}[\delta(h^*+y+z)-\delta(h^*+z)]dzdy\\
&=q^{-1}\sum_{k=1,2}a_k^+\mathcal{D}_k^+e^{\beta_k^+(x-h^*)},
\end{aligned}$$

where the constants

$$\mathcal{D}_k^+=\sum_{j=1,2}(-a_j^-)\int_0^{+\infty}\int_0^{+\infty}e^{\beta_j^- z-\beta_k^+ y}[\delta(h^*-z)-\delta(h^*+y-z)]dzdy$$

can be easily calculated in Examples 1–5 of Section 2, and in many other examples. The calculations for the put options are similar.

6 Proof of the main results for the perpetual put

We use the following general lemma and theorem (Lemma 5.1, Theorem 2.12 and Remark 2.1 in Boyarchenko and Levendorskiǐ, 2002b; Lemma 7.1 and Theorem 2.1 in Boyarchenko and Levendorskiǐ, 2002b). We formulate them for a special case of a process on the line, and an optimal stopping region of the form $(-\infty, h]$.

Lemma 6.1. *Let h_* and a function V_* satisfy the following conditions:*

$$(q-L)V_*(x) = 0, \quad x > h_*; \tag{6.62}$$
$$V_*(x) = G(x), \quad x \le h_*; \tag{6.63}$$
$$V_*(x) \ge \max\{G(x), 0\}, \quad x \in \mathbf{R}; \tag{6.64}$$
$$(q-L)V_*(x) \ge 0, \quad x < h_*; \tag{6.65}$$
$$W_* := (q-L)V_* \quad \text{is universally measurable;} \tag{6.66}$$
$$U^q_X W_* = V_*. \tag{6.67}$$

Then $\tau^-_{h_*}$, is the optimal stopping time for (2.6) in the class $\mathcal{M}$, and V_* is the rational put price.

The first four conditions have the natural economic meaning, and the last two are technical regularity conditions. Equations (6.62) and (6.65) mean that in the optimal inaction region, the instantaneous expected return equal the riskless rate, and in the action region, the latter becomes larger than the former. Equation (6.63) states that at the moment of exercise, the option price equals the payoff, and (6.64) means that the option value cannot be negative, and it cannot be less than the payoff. It is interesting that the natural optimality conditions are easy to verify if the payoff is modeled in the natural fashion, as the EPV of a monotone stream.

Theorem 6.2. *Let X_t satisfy the (ACP)-property, and let G be a continuous function satisfying (4.35). Then for any h_*, $V_0^-(h_*;x) := E^x\left[e^{-q\tau^-_h} G(X_{\tau^-_h})\right]$ is a unique solution of the problem (6.62)–(6.63) in the class of function satisfying (4.35) and continuous on $(h_*, +\infty)$.*

It remains to show that if h_* is the solution to (4.42), then $V_*(x) := V_0^-(h_*;x)$ satisfies (6.64)–(6.67). A relatively short and simple verification below is based on the following theorem.

Theorem 6.3. *Let X_t satisfy the (ACP)-property, and let g be measurable, locally bounded, non-negative, and satisfy (4.35). Then*

$$E^x\left[e^{-q\tau^-_h} U^q_X g(X_{\tau^-_h})\right] = qU^q_{\underline{X}} \mathbf{1}_{(-\infty,h]} U^q_{\bar{X}} g(x), \quad \forall\, x > h. \tag{6.68}$$

Proof. This theorem was proved in Boyarchenko and Levendorskiǐ (2002a,b) for wide classes of Lévy processes. In Section 9, we give the proof for a much wider class of Lévy processes satisfying the (ACP)-property. ∎

Set $V^-(h_*;x) := qU^q_{\underline{X}}\mathbf{1}_{(-\infty,h_*]}U^q_{\bar{X}}g(x)$.

Corollary 6.4. *For any x, $V_*(x) = V^-(h_*;x)$.*

Proof. By Theorem 6.3, the equality holds for $x > h$, and since V_* satisfies (6.63), it suffices to show that $V^-(h_*;x) = G(x)$ for $x \le h_*$ as well. We use (5.53) to represent $V^-(h_*;x)$ in the form

$$\begin{aligned} V^-(h_*;x) &= qU^q_{\underline{X}}U^q_{\bar{X}}g(x) - q(U^q_{\underline{X}}\mathbf{1}_{(h_*,+\infty)}U^q_{\bar{X}}g)(x) \\ &= U^q_X g(x) - v(h_*;x) = G(x) - v(h_*;x), \end{aligned} \tag{6.69}$$

where $v(h_*;x) = q(U^q_{\underline{X}}\mathbf{1}_{(h_*,+\infty)}U^q_{\bar{X}}g)(x)$. For $x \le h_*$,

$$v(h_*;x) = qE\left[\int_0^{+\infty} e^{-qt}(\mathbf{1}_{(h_*,+\infty)}U^q_{\bar{X}}g)(\underline{X}_t)dt \mid X_0 = x\right] = 0,$$

therefore $V^-(h_*;x) = G(x), x \le h_*$. ∎

Verification of (6.64). We assumed that g is continuous or X satisfies the (CR)-condition, hence $w := U^q_{\bar{X}}g$ is continuous, and h_*, the solution to the equation (4.42), exists, and it is unique. It follows that $\mathbf{1}_{(-\infty,h]}w$ is continuous if and only if $h = h_*$, therefore this choice of the exercise boundary makes $V^-(h;x)$ maximally regular. This observation extends the smooth pasting principle as the informal principle for the choice of the optimal exercise boundary. Due to the choice of h_*, $\mathbf{1}_{(-\infty,h_*]}w$ is continuous and non-negative. Therefore, $V_*(\cdot) = V^-(h_*;\cdot)$ is continuous, and

$$\begin{aligned} V^-(h_*;x) &= qU^q_{\underline{X}}\mathbf{1}_{(-\infty,h_*]}w(x) \\ &= qE\left[\int_0^{+\infty} e^{-qt}\left(\mathbf{1}_{(-\infty,h_*]}w\right)(x + \underline{X}_t)dt\right] \ge 0, \quad \forall\, x. \end{aligned} \tag{6.70}$$

Since w in non-increasing, $\mathbf{1}_{(h_*,+\infty)}w$ and $v(h_*;\cdot)$ in (6.69) are non-positive, and we conclude from (6.69) that $V^-(h_*;x) \ge G(x)$ for all x. Thus, V_* satisfies (6.64).
Verification of (6.65)–(6.67). Since (6.62) holds, we need to check (6.65) on $(-\infty,h_*)$. Below, we will show that

$$W_* = (q-L)V_* \quad \text{is non-increasing on } (-\infty,h_*), \tag{6.71}$$

and

$$W_*(h_* - 0) \ge 0. \tag{6.72}$$

Conditions (6.65) and (6.66) follow immediately from (6.71) and (6.72), and it remains to check (6.67). Since $\mathbf{1}_{(-\infty,h_*]}U^q_{\bar{X}}g$ is continuous, $V_* = V^-(h_*;\cdot)$ given by the LHS in (6.68) is continuous, and since W_* is universally measurable and X

satisfies the (ACP)-property, $U^q_X W_*$ is continuous. Therefore it suffices to prove (6.67) in the sense of generalized functions:

$$\int_{-\infty}^{+\infty} U^q_X W_*(x)u(x)dx = \int_{-\infty}^{+\infty} V_*(x)u(x)dx, \quad \forall\, u \in C_0^\infty(\mathbf{R}). \tag{6.73}$$

By the standard duality argument,

$$\int_{-\infty}^{+\infty} U^q_X (q-L)V_*(x)u(x)dx = \int_{-\infty}^{+\infty} V_*(x)(q-\tilde{L})U^q_{\tilde{X}} u(x)dx,$$

where $\tilde{X}$ is the dual process and $\tilde{L}$ its generator. Since $u \in C_0^\infty(\mathbf{R})$, $(q-\tilde{L})U^q_{\tilde{X}} u = u$, hence (6.73) holds, and the proof of (6.67) is finished.

Proof of (6.71) and (6.72). Represent W_* in the form

$$W_* = (q-L)G - (q-L)w^1,$$

where $w^1 = qU^q_{\underline{X}} \mathbf{1}_{(h_*,+\infty)} w$. Since supp $w^1 \subset [h_*, +\infty)$, and the Gaussian part of the infinitesimal generator is a local (differential) operator, we have for $x < h_*$

$$\begin{aligned} W_*(x) &= (q-L)G(x) \\ &\quad + \int_{-\infty}^{+\infty} \left(w^1(x+y) - w^1(x) - \mathbf{1}_{[-1,1]}(y)(w^1)'(x) \right) F(dy) \\ &= g(x) + \int_{h_*-x}^{+\infty} w(x+y)F(dy), \end{aligned}$$

where $F(dy)$ is the Lévy density. Since $g = (q-L)G$ and $w = U^q_{\bar{X}} g$ are non-increasing, both terms on the RHS are non-increasing, and (6.71) is proved.

Finally, assume that (6.72) fails. On the strength of (6.71), W_* must be negative on some interval (h, h_*), where $h < h_*$. By applying $U^q_{\bar{X}}$ to $W_* = (q-L)V_*$ and using (6.68) and (5.53), we obtain

$$U^q_{\bar{X}} W_* = U^q_{\bar{X}} (q-L) q U^q_{\underline{X}} \mathbf{1}_{(-\infty,h_*]} w = \mathbf{1}_{(-\infty,h_*]} w.$$

For $x \in (h, h_*)$, we have $U^q_{\bar{X}} W_*(x) \le 0$, but $\mathbf{1}_{(-\infty,h_*]} w(x) > 0$, a contradiction. Thus, (6.72) holds, and the proof of the optimality of the stopping time $\tau^-_{h_*}$ has been completed. ∎

7 Proof of the main results for the perpetual call

We assume that the supremum process is non-trivial, and g is piece-wise continuous, and it satisfies (4.35). Since X satisfies the (ACP)-property, $G = U^q_X g$ is continuous, and it satisfies (4.35).

Fix $h \in \mathbf{R}$, and set $\tau = \tau^+_h = \inf\{\, t \mid X_t > h\}$. If we change the direction on the real line, a neighborhood of $-\infty$ becomes a neighborhood of $+\infty$, the supremum process becomes the infimum process, and vice versa. Hence, by symmetry, we obtain an analog of Theorem 6.3.

Theorem 7.1. *Let X_t satisfy the (ACP)-property, and let g be measurable, locally bounded, non-negative, and satisfy (4.35). Then*

$$E^x\left[e^{-q\tau_h^+}U_X^q g(X_{\tau_h^+})\right] = qU_{\bar{X}}^q \mathbf{1}_{[h,+\infty)}U_{\underline{X}}^q g(x), \quad \forall\, x<h. \tag{7.1}$$

The proof of the optimality of the early exercise boundary (4.39) is the mirror reflection of the proof for the put option.

8 Carr's randomization

In the finite time horizon case, exact formulas for the early exercise boundary are not available even in the Brownian motion case. There are several approximate methods – see, e.g., the discussion and references in Boyarchenko and Levendorskiĭ (2004). An approximate method for a finite horizon problem is based on the time discretization. One of the first methods of this kind is the analytical method of lines suggested by Carr and Faguet (1994); another interpretation of the same pricing procedure was given by Carr (1998). Carr and Faguet (1994) and Carr (1998) considered the Brownian motion case. Boyarchenko and Levendorskiĭ (2002b, Chapter 6) generalized this method for wide classes of Lévy processes, and in Levendorskiĭ (2004), an efficient pricing procedure for the put under exponential jump-diffusion processes was suggested. A different generalization of Carr's randomization method for spectrally negative Lévy processes was used in Avram et al. (2002).

In this section, we generalize results in Boyarchenko and Levendorskiĭ (2002b) and Levendorskiĭ (2004). We consider both calls and puts with payoffs of the form $G=U_X^r g$, where $r>0$ is the riskless rate (in this section, it is convenient to use r, and reserve the label q for other purposes). On each step of the method, we reduce the problem of the choice of the approximation to the early exercise boundary to an optimal stopping problem for a call-like or put-like option, with an auxiliary payoff. A piece of good luck is that on each step, the auxiliary payoff is monotone (this is proved by the mathematical induction argument), and therefore the same argument as in Section 6 and Section 7 apply.

Call option. We assume that a stream g is non-decreasing, and satisfies (4.38) and (4.35), and either g is continuous or the process X_t satisfies the (CR)- condition discussed at the end of Section 2.

Let T be the maturity date. We divide the period $[0,T]$ into n sub-periods of length $\Delta=T/n$, set $t_j=j\Delta$, and find approximations h_j and v_j to the optimal exercise boundary and option value at time t_j using the backward induction, as follows. At time T, $v_n(x)=G(x)^+$. For $j=n-1,n-2,\dots,0$, the function v_j solves the time-discretized version of the generalized Black-Scholes equation on $[t_j,t_{j+1}]$, and the early exercise boundary is determined so as to make $v_j(x)$ maximal. For the call option, the problem is

$$\Delta^{-1}(v_{j+1}(x)-v_j(x))+(L-r)v_j(x) = 0, \quad x<h_j, \tag{8.75}$$

$$v_j(x) = G(x), \quad x\geq h_j. \tag{8.76}$$

In (8.76), we may write $G(x)$ instead of $G(x)^+$, since it is non-optimal to exercise the option at negative levels of the payoff. The sequence of problems (8.75)–(8.76) is the analytical method of lines, and Carr's randomization is a less formal argument, which yields the same sequence. For details, see Chapter 6 in Boyarchenko and Levendorskiǐ (2002b).

The backward induction procedure for the call option is:

I. Define $v_n(x) = G(x)^+$, $q = \Delta^{-1} + r$, and calculate $U^r_X g$ and

$$\tilde{g} = (q - L)G = (q - L)U^r_X g = (\Delta^{-1} + r - L)U^r_X g = g + \Delta^{-1}U^r_X g.$$

II. For $j = n-1, n-2, \dots, 0$, calculate

$$z_j = v_{j+1} - \Delta \cdot \tilde{g}; \tag{8.77}$$
$$w_j = U^q_{\underline{X}} z_j; \tag{8.78}$$
$$h_j, \quad \text{the zero of } w_j; \tag{8.79}$$
$$v_j = \Delta^{-1} q U^q_{\bar{X}} \mathbf{1}_{(-\infty, h_j)} U^q_{\underline{X}} z_j + U^r_X g. \tag{8.80}$$

Notice that G and $\tilde{g}$ satisfy the same conditions (4.38), (4.35) and monotonicity/continuity conditions as g, and that in the process of the proof, it will be shown that a zero exists, and it is unique. We rewrite (8.75)–(8.76) in the form

$$(q - L)v_j(x) = \Delta^{-1} v_{j+1}(x), \quad x < h_j, \tag{8.81}$$
$$v_j(x) = U^q_X \tilde{g}(x), \quad x \ge h_j. \tag{8.82}$$

Introduce $V_j = v_j - \Delta^{-1} U^q_X v_{j+1}$. Since $(q - L)U^q_X = I$, V_j solves (8.81)–(8.82) with 0 and $\tilde{g} - \Delta^{-1} v_{j+1}$ in the RHS, respectively. Since X_t satisfies the (ACP)-condition, the continuous solution to (8.81)–(8.82), which is bounded on $(-\infty, h_j]$, is

$$V_j = E^x \left[e^{-q\tau^+_{h_j}} U^q_X (\tilde{g} - \Delta^{-1} v_{j+1})(X(\tau^+_{h_j})) \right]$$

(This follows from the mirror relection of Theorem 6.2). Theorem 7.1 gives

$$\begin{aligned} v_j &= q U^q_{\bar{X}} \mathbf{1}_{[h_j, +\infty)} U^q_{\underline{X}} (\tilde{g} - \Delta^{-1} v_{j+1}) + \Delta^{-1} U^q_X v_{j+1} \\ &= q U^q_{\bar{X}} \mathbf{1}_{(-\infty, h_j)} U^q_{\underline{X}} (\Delta^{-1} v_{j+1} - \tilde{g}) \\ &\quad + q U^q_{\bar{X}} U^q_{\underline{X}} (\tilde{g} - \Delta^{-1} v_{j+1}) + \Delta^{-1} U^q_X v_{j+1} \\ &= q U^q_{\bar{X}} \mathbf{1}_{(-\infty, h_j)} U^q_{\underline{X}} (\Delta^{-1} v_{j+1} - \tilde{g}) + U^r_X g, \end{aligned}$$

since $qU^q_{\bar{X}} U^q_{\underline{X}} = U^q_X$, and $U^q_X \tilde{g} = U^q_X (q - L)U^r_X g = U^r_X g$. Define $z_j = v_{j+1} - \Delta \cdot \tilde{g}$, and write v_j in the form (8.80). For $j = n-1$, $v_{j+1} = (U^r_X g)^+$. Hence, by using the equalities $\tilde{g} = (I + \Delta^{-1} U^r_X) g$ and $\Delta(q - r) = 1$, we obtain

$$z_{n-1} = (U^r_X g)^+ - \Delta \cdot \tilde{g} = (U^r_X g)^+ - U^r_X g - \Delta \cdot g = (U^r_X g)^- - \Delta \cdot g,$$

where $(U^r_X g)^-(x) = \min\{0, U^r_X g(x)\}$. Due to the conditions imposed on g, z_{n-1} satisfies the following conditions

$$z_j \quad \text{is non} - \text{increasing, and continuous if } g \text{ is continuous;} \tag{8.83}$$

$$z_j(-\infty) > 0, \quad z_j(+\infty) < 0; \tag{8.84}$$

$$|z_j(x)| \le C(e^{\sigma_+ x} + e^{\sigma_- x}). \tag{8.85}$$

Set $w_j = U^q_{\underline{X}} z_j$. Clearly, if z_j satisfies (8.83)–(8.85), then w_j does, and moreover, due to our assumption on g and X, w_j is continuous. Hence, the equation

$$w_j(x) = 0 \tag{8.86}$$

has a zero, call it h_j. On the strength of (8.84), w_j may be locally constant only in a neighborhood of $+\infty$, where it is negative. Hence, the zero is unique. So far, we have established these properties of z_j and w_j for $j = n-1$. This is the basis for our backward induction.

Lemma 8.1. *Assume that at step j, z_j satisfies (8.83)–(8.85). Then h_j is the optimal exercise boundary at time t_j.*

Proof. The proof above shows that w_j is a non-increasing continuous function, which changes sign only once; hence, h_j is well-defined. Denote by $v_j(x, h_j)$ the solution to (8.81)–(8.82). We compare $v_j(x, h_j)$ and $v_j(x, h'_j)$. They are given by (8.80) with different boundaries h_j and h'_j but the same z_j. Suppose that $h'_j < h_j$. Then

$$v_j(x, h_j) - v_j(x, h'_j) = \Delta^{-1} q U^q_{\bar{X}} \mathbf{1}_{[h'_j, h_j)} U^q_{\underline{X}} z_j(x). \tag{8.87}$$

Since $w_j = U^q_{\underline{X}} z_j$ is positive on $(-\infty, h_j)$, the RHS in (8.87) is non-negative for all x, and positive for some $x < h_j$. Hence, the choice $h'_j < h_j$ is non-optimal. Similarly, if $h'_j > h_j$, then

$$v_j(x, h'_j) - v_j(x, h_j) = \Delta^{-1} q U^q_{\bar{X}} \mathbf{1}_{[h_j, h'_j)} U^q_{\underline{X}} z_j(x). \tag{8.88}$$

Since $w_j = U^q_{\underline{X}} z_j$ is negative on $(h_j, +\infty)$, the RHS in (8.87) is non-negative for all x, and positive for some $x < h'_j$. Hence, the choice $h'_j < h_j$ is non-optimal. ∎

We conclude that h_{n-1} is the optimal exercise boundary on step $n-1$. Now it remains to prove that if (8.83)–(8.85) hold for $j = n-1, n-2, \dots, l+1$, and h_j is chosen as the solution to (8.86), $j = n-1, n-2, \dots, l+1$, then z_l satisfies (8.83)–(8.85) as well. Using (8.80) and the equality

$$U^r_X g - \Delta \cdot \tilde{g} = U^r_X g - \Delta \cdot (g + \Delta^{-1} U^r_X g) = -\Delta \cdot g,$$

we obtain

$$\begin{aligned} z_l &= v_{l+1} - \Delta \cdot \tilde{g} \\ &= \Delta^{-1} q U^q_{\bar{X}} \mathbf{1}_{(-\infty,h_{l+1})} U^q_{\underline{X}} z_{l+1} + U^r_X g - \Delta \cdot \tilde{g} \\ &= \Delta^{-1} q U^q_{\bar{X}} \mathbf{1}_{(-\infty,h_{l+1})} U^q_{\underline{X}} z_{l+1} - \Delta \cdot g. \end{aligned}$$

Due to the choice of h_{l+1}, the function $\mathbf{1}_{(-\infty,h_{l+1})} U^q_{\underline{X}} z_{l+1}$ satisfies conditions (8.83)–(8.85), with the only exception: the limit at $+\infty$ is 0. Since $\Delta > 0$, and $-g$ satisfies (8.83)–(8.85), we conclude that z_l satisfies (8.83)–(8.85).

Put option. We assume that a stream g is non-increasing, and satisfies (4.41) and (4.35), and either g is continuous or the process X_t satisfies the (PR)- condition discussed at the end of Section 2. By changing the direction on the real line, we transform the pricing procedure for the call into the pricing procedure for the put; the infimum process becomes the supremum process, and vice versa:

I. Define $v_n(x) = G(x)^+$, $q = \Delta^{-1} + r$, and calculate $U^r_X g$ and

$$\tilde{g} = (q - L)G = (q - L)U^r_X g = (\Delta^{-1} + r - L)U^r_X g = g + \Delta^{-1} U^r_X g.$$

II. For $j = n-1, n-2, \dots, 0$, calculate

$$\begin{aligned} z_j &= v_{j+1} - \Delta \cdot \tilde{g}; \\ w_j &= U^q_{\bar{X}} z_j; \\ h_j, &\quad \text{the zero of } w_j; \\ v_j &= \Delta^{-1} q U^q_{\underline{X}} \mathbf{1}_{(h_j,+\infty)} U^q_{\bar{X}} z_j + U^r_X g. \end{aligned}$$

9 Proof of Theorem 6.3

Fix $h \in \mathbf{R}$, and set $\tau = \tau^-_h = \inf\{\, t \mid X_t < h\}$. For z in the upper right-plane $\Re z \geq 0$, define functions $u(z;x) = u(h,z;x) = \mathbf{1}_{(-\infty,h]}(x) e^{zx}$, and

$$\begin{aligned} f(z;x) &= f(q,h,z;x) = E^x[\exp(zX_\tau - q\tau)], \\ f^1(z;x) &= f^1(q,h,z;x) = \kappa^-_q(z)^{-1} q U^q_{\underline{X}} u(z;x). \end{aligned}$$

Lemma 9.1. *For $x > h$, $f(z;x) = f^1(z;x)$.*

Proof. For a fixed x, both functions are analytic in the half-plane $\Re z > 0$, and continuous up to the boundary. Hence, it suffices to prove the equality for $z \in [0, \epsilon]$, where ε is some positive number. If $z = 0$, then $\kappa^-_q(z)^{-1} = 1$, and the equality

$$E^x[\exp(zX_\tau - q\tau)] = qE^x\left[\int_0^{+\infty} e^{-qt} \mathbf{1}_{(-\infty,h]}(\underline{X}_t) dt\right]$$

holds. Thus, the lemma is proved in the case $z = 0$. The proof for small positive z consists of the following steps: $f(z;\cdot)$ is RCLL on $(h;+\infty)$ (right continuous with left limits); $f^1(z;\cdot)$ is RCLL on $(h;+\infty)$; the Laplace transforms of these two functions are equal.

Function $f(0;\cdot)$ is q-excessive (Proposition 41.5 (ii) and (viii) in Sato (1999)). Since X satisfies the (ACP)-property, a q-excessive function is lower semi-continuous (Theorem 41.5 (4) in Sato (1999)), but $f(0;\cdot)$ is evidently non-increasing; hence, $f(0;\cdot)$ and $f^1(0;\cdot)$ are RCLL on $(h,+\infty)$. Consider sufficiently small $z>0$ so that $q-\Psi(z)>0$. Introduce $\Psi_z(w):=\Psi(w+z)-\Psi(z)$. This is the Lévy exponent of X_t under a generalization of the Esscher transform of the measure $\mathbf{Q}$ (the Esscher transform uses a particular value z, which makes e^{-qt+X_t} a local martingale); denote this transform by $\mathbf{Q}_z$. Let $E^{\mathbf{Q}}$ and $E^{\mathbf{Q}_z}$ be the expectation operators under $\mathbf{Q}$ and $\mathbf{Q}_z$, respectively. We have

$$f(z;x):=E^{\mathbf{Q};x}[\exp(-q\tau_h^- + zX_{\tau_h^-})]=e^{zx}E^{\mathbf{Q}_z;0}[\exp(-(q-\Psi(z))\tau_h^-)].$$

Since X satisfies the (ACP)-property under $\mathbf{Q}$, it satisfies the (ACP) property under $\mathbf{Q}_z$. Hence, the last factor on the RHS is RCLL on $(h,+\infty)$, and $f(z;\cdot)$, its product with a continuous function, is RCLL on $(h,+\infty)$ as well.

To prove that $f^1(z;\cdot)$ is RCLL on $(h,+\infty)$, we change the variables $x\mapsto x+h$, so that h becomes 0, and represent $f^1(z;\cdot)$ in the form

$$f^1(z;x)=\kappa_q^-(z)^{-1}\left(qU^q_{\underline{X}}\mathbf{1}_{(-\infty,0]}(x)+qU^q_{\underline{X}}\mathbf{1}_{(-\infty,0]}(e^{z\cdot}-1)(x)\right), \quad (9.89)$$

The function $\mathbf{1}_{(-\infty,0]}(x)(e^{zx}-1)$ is continuous, therefore the second term in the brackets on the RHS in (9.89) is continuous. The first term in the brackets equals $f(0;x)$, hence it is RCLL on $(h,+\infty)$. We conclude that the sum is RCLL on $(h,+\infty)$.

Now we consider the Laplace transforms. The fluctuation identity (3.13) in Hilberink–Rogers (2002) states that for any $\mu>z$,

$$\int_0^\infty e^{-\mu x}f(z;x)dx=\frac{1}{\mu-z}\left[1-\frac{\kappa_q^-(\mu)}{\kappa_q^-(z)}\right],$$

and therefore we need to prove the same equality for f^1. We have

$$f^1(z;x)=\kappa_q^-(z)^{-1}qU^q_{\underline{X}}\left(1-\mathbf{1}_{(0,+\infty)}(x)\right)e^{zx}=e^{zx}-\kappa_q^-(z)^{-1}qU^q_{\underline{X}}v(z;x),$$

where $v(z;x)=\mathbf{1}_{(0,+\infty)}(x)e^{zx}$. For w in the upper half-plane $\Re w>\Re z$,

$$\hat{v}(z;w)=\int_0^{+\infty}e^{-wx}v(z;x)dx=\int_0^{+\infty}e^{-x(w-z)}dx=(w-z)^{-1}$$

is well-defined; therefore, using the inverse Laplace transform, we obtain

$$f^1(z;x)=e^{zx}-(2\pi i)^{-1}\int_{-i\infty+\sigma}^{+i\infty+\sigma}e^{wx}\kappa_q^-(z)^{-1}\kappa_q^-(w)(w-z)^{-1}dw, \quad (9.90)$$

for any $\sigma > \Re z$. Take $\mu > z$ (and $\sigma \in (z, \mu)$), and calculate the Laplace transform

$$\int_0^{+\infty} e^{-\mu x} f^1(z;x)dx = \frac{1}{\mu - z} - \frac{\kappa_q^-(\mu)}{\kappa_q^-(z)(\mu - z)} = \frac{1}{\mu - z}\left[1 - \frac{\kappa_q^-(\mu)}{\kappa_q^-(z)}\right].$$

Thus, the Laplace transforms of $f(z;\cdot)$ and $f^1(z;\cdot)$ are equal. ■

Proof of Theorem 6.3. If the infimum process is trivial, both sides of (6.68) are 0, so we assume that it is non-trivial. Consider first $g(x) = e^{zx}$, where $z \in i\mathbf{R}$. Using (5.48) and (5.49), we can rewrite (6.68) as $(q - \Psi(z))^{-1} f(z;x) = q^{-1}\kappa_q^+(z) f^1(z;x)$, where f and f^1 are the functions in Lemma 9.1. This equality holds on the strength of (5.53) and Lemma 9.1, hence (6.68) is proved for oscillating exponents.

Next, consider $g \in C_0^\infty(\mathbf{R})$. We represent g as the Fourier integral, use Lemma 9.1 under the integral sign, and obtain (6.68). Finally, an arbitrary non-negative locally bounded measurable function can be approximated (point-wise) by a non-decreasing sequence of compactly supported locally bounded measurable functions. Using the Dominant Convergence Theorem, we conclude that it suffices to consider a compactly supported g. Such a g can be approximated in L^∞-topology by indicator functions of open bounded sets, hence, it suffices to consider these indicator functions. An indicator function of an open set can be approximated by a non-decreasing sequence of functions of the class $C_0^\infty(\mathbf{R})$, therefore it remains to prove Theorem 6.3 for these functions, which we have done already. ■

10 Conclusion

In the paper, we considered optimal stopping problems in infinite time horizon case, when the underlying process is a process with stationary i.i.d. increments (Lévy process). Contrary to the bulk of the optimal stopping literature, which paid most attention to the generality of a process but treated only the simplest payoff functions, we develop an analytically tractable method, which can be used to price options with payoffs of a very general class. The explicit results are obtained under an economically meaningful condition that the option price is the expected present value (EPV) of a stream of profits/dividends, which is a monotone function of the underlying process. We show that using an appropriate functional dependence, it is possible to obtain analytically tractable results for payoff processes, which exhibit mean-reverting, stochastic volatility and/or switching features. For the case of the Brownian motion with embedded exponentially distributed jumps, we derived simple equations for the optimal exercise threshold, and formulas for the option price. We extended the interpretation of the optimal exercise rules as *record setting news principles* to wider classes of payoff streams than in Boyarchenko (2004).

Finally, we developed a semi-explicit pricing procedure for Carr's randomization approximation to American options with finite time horizon.

References

Alili, L., Kyprianou, A.E.: Some remarks on first passage of Lévy processes, the American put and pasting principles. Working paper (2004)

Avram, F., Chan, T., Usabel, M.: On the valuation of constant barrier options under spectrally one-sidedexponential Lévy models and Carr's randomization for American puts. Stochastic Processes and their Applications **100**, 75–107 (2002)

Barndorff-Nielsen, O.E.: Processes of normal inverse Gaussian type. Finance and Stochastics **2**, 41–68 (1998)

Bates, D.S.: Empirical option pricing: a retrospection. Journal of Econometrics **116**, 387–404 (2003)

Bernanke, B.S.: Irreversibility, uncertainty, and cyclical investment. Quarterly Journal of Economics **98**, 85–106 (1983)

Bertoin, J.: Lévy processes Cambridge tracts in mathematics, vol. 121. Cambridge: Cambridge University Press 1996

Boyarchenko, S.I.: Irreversible decisions and record setting news principles. American Economic Review **94**(3), 557–568 (2004)

Boyarchenko, S.I., Levendorskiĭ, S.Z.: Option pricing for truncated Lévy processes. International Journal of Theoretical and Applied Finance **3**, 549–552 (2000)

Boyarchenko, S.I., Levendorskiĭ, S.Z.: Perpetual American options under Lévy processes. SIAM Journal on Control and Optimization **40**, 1663–1696 (2002a)

Boyarchenko, S.I., Levendorskiĭ, S.Z.: Non-Gaussian Merton-Black-Scholes theory. Singapore: World Scientific 2002b

Boyarchenko, S.I., Levendorskiĭ, S.Z.: Barrier options and touch-and-out options under regular Lévy processes of exponential type. Annals of Applied Probability **12**, 1261–1298 (2002c)

Boyarchenko, S.I., Levendorskiĭ, S.Z.: Practical guide to real options in discrete time. International Economic Review (submitted) (http://ssrn.com/abstract=510324) (2004)

Carr, P.: Randomization and the American put. Review of Financial Studies **11**, 597–626 (1998)

Carr, P., Faguet, D.: Fast accurate valuation of American options. Working paper, Cornell University, Ithaca (1994)

Carr, P., Geman, H., Madan, D.B., Yor, M.: The fine structure of asset returns: an empirical investigation. Journal of Business **75**, 305-332 (2002)

Chernov, M., Gallant, A.R., Ghysels, E., Tauchen, G.: Alternative models for stock price dynamics. Journal of Econometrics **16**, 225–257 (2003)

Darling, D.A., Liggett, T., Taylor, H.M.: Optimal stopping for partial sums. Annals of Mathematical Statistics **43**, 1363–1368 (1972)

Dixit, A.K., Pindyck, R.S.: Investment under uncertainty. Princeton: Princeton University Press 1996

Duffie, D., Pan, J., Singleton, K.: Transform analysis and option pricing for affine jump-diffusions. Econometrica **68**, 1343–1376 (2000)

Eberlein, E., Keller, U., and Prause, K.: New insights into smile, mispricing and value at risk: the hyperbolic model. Journal of Business **71**, 371–406 (1998)

Gerber, H.U., Shiu, E.S.W.: Martingale approach to pricing perpetual American options on two stocks. Mathematical Finance **6**, 303–322 (1996)

Hilberink, B., and Rogers, L.C.G.: Optimal capital structure and endogenous default. Finance and Stochastics **6**, 237–263 (2002)

Koponen, I.: Analytic approach to the problem of convergence of truncated Lévy flights towards the Gaussian stochastic process. Physics Review E **52**, 1197–1199 (1995)

Levendorskiĭ, S.: Pricing of American put under Lévy processes. International Journal of Theoretical and Applied Finance **7**, 303–335 (2004)

Mc Kean, H. P., Jr.: Appendix: a free boundary problem for the heat equation arising from a problem in Mathematical economics. Industrial Management Review **6**, 32–39 (1965)

Merton, R. C.: The theory of rational option pricing. Bell Journal of Economics **4**, 141–183 (1973)

Mordecki, E.: Optimal stopping and perpetual options under Lévy processes. Finance and Stochastics **6**, 473–493 (2002a)

Mordecki, E.: Perpetual options for Lévy processes in the Bachelier model. Proceedings of the Steklov Mathematical Institute **237**, 256–264 (2002b)

Mordecki, E.: A note on "Optimal stopping and perpetual options under Lévy processes". Working paper, May 11 (2004) http//:www.cmat.edu.uy/ mordecki/articles/note

Sato, K.: Lévy processes and infinitely divisible distributions. Cambridge: Cambridge University Press 1999